PROBABILITY CURRENT IN ZERO-SPIN RELATIVISTIC QUANTUM MECHANICS

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Klein–Gordon equation for scalar wave function:

no conserving positive definite probability current

only conserving charge current

Feshbach–Villars formalism on $\begin{pmatrix} \varphi + \frac{\mathrm{i}}{m}\left(\partial_0 + \mathrm{i}eA_0\right)\varphi \\ \varphi - \frac{\mathrm{i}}{m}\left(\partial_0 + \mathrm{i}eA_0\right)\varphi \end{pmatrix}$

efforts to choose the 'positive' solutions

Spin 1/2: $\overline{\psi}\gamma^\mu\psi$, no efforts needed

$$(154) \qquad (U_{h,x}\varphi)(p) = \exp i\{x,p\}\varphi(\delta(h)^{-1}p)^h.$$

$U \simeq U^{m,+,j}$ by theorem 6.20.

The reader might have noticed that we have not treated the spinless case in terms of the spinor calculus. This can also be done provided we consider, instead of the bundle (149), the bundle where the fibers are made up of *skew symmetric* tensors. More precisely, let

$$(155) \qquad B_m^{+,0} = \{(p,t) : p \in X_m^+, t \in \mathbf{C}^4 \otimes \mathbf{C}^4, t \text{ skew symmetric}, (\textstyle\sum p_\nu \gamma_\nu^{\nu})t = mt, \nu = 1, 2\},$$

with the norm defined for any Borel section φ of the bundle by

$$(156) \qquad \|\varphi\|^2 = \int_{R^3} \langle \varphi(p), \varphi(p) \rangle_2 \, d\beta_m^{+,2}(\mathbf{p}).$$

The representation of G^* is then defined in the same way as (154). For a given $p \in X_m^+$, the fiber of $B_m^{+,0}$ at p consists of all skew symmetric elements of $B_m^{+,1/2}(p) \otimes B_m^{+,1/2}(p)$ which form an one-dimensional space. As the unitary group K^* has no nontrivial one-dimensional representations,

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it induces the trivial representation on the fiber of $B_m^{+,0}$ at $(m,0,0,0)$. Hence the representation of G^* defined in the Hilbert space of sections of $B_m^{+,0}$ is equivalent to $U^{m,+,0}$.

The Representations $U^{+,\pm n}$. These representations may be obtained by passing to the limit in $U^{m,+,j}$ as $m \to 0+$. We write

Varadarajan: Geometry of quantum theory I–II. (1968–70) only the free case

only in momentum space

only the representation

coordinate space?

physically interesting investigations?

nonfree extension?

(first of all: how does his construction work?)

geometric picture for the spin 1/2 case $\Longrightarrow$ spin 0 case easy

Notations:

$\hbar \equiv c \equiv 1$,

g with signature $(+---)$,

spacetime four-indices: $\mu, \nu = 0, 1, 2, 3$

three-indices: $j, k = 1, 2, 3$

complex conjugate: *, transposition: $^{\mathrm{T}}$, combination: †

$m > 0$ only, $p \in \mathcal{P}_m$: $p_0 = \sqrt{p_j p_j + m^2}$, $\int_{\mathcal{P}_m} \frac{\mathrm{d}^3 p}{p_0}$

Matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\beta = \gamma_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

Properties:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}, \qquad \gamma_0^\dagger = \gamma_0, \qquad \gamma_j^\dagger = -\gamma_j,$$

$$\alpha_j = -\beta \gamma_j, \qquad \gamma_\mu^\dagger \beta = \beta \gamma_\mu, \qquad \alpha_j^\dagger = \alpha_j$$

Frequently: $\theta \in \mathbb{C}^4$, $\Theta \in \mathbb{C}^4 \otimes \mathbb{C}^4$: $\qquad \overline{\theta} = \theta^\dagger \beta, \quad \overline{\Theta} = \Theta^\dagger \beta$

Geometric ingredients:

$p \in \mathcal{P}_m$: eigenvalues of $p_\mu \gamma^\mu$ are $+m$, $-m$,

the eigensubspaces: $N_+(p)$, $N_-(p)$, both 2 dimensional

For example: $\check{p} = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$: $\check{p}_\mu \gamma^\mu = m\gamma^0 = m\beta$,

$$N_+(\check{p}): \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad N_-(\check{p}): \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\alpha_j p_j + \beta m$: $+p_0$, $-p_0$, $N_+(p)$, $N_-(p)$

Spin 1/2, free quantum mechanics:

$\gamma^\mu p_\mu \psi(p) = m\psi(p)$: $\qquad \psi(p) \in N_+(p)$

For example: $\quad \check{p} = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$: $\quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix}$

To Lorentz transformation L, exists D_L:

$$\beta D_L^\dagger \beta = D_L^{-1}, \qquad D_L \left(\gamma^\mu p_\mu\right) D_L^{-1} = \gamma^\mu (Lp)_\mu$$

$\psi : \mathcal{P}_m \to \mathbb{C}^4$: $\qquad \left(U_{(a,L)}\psi\right)(p) = \mathrm{e}^{\mathrm{i}p_\mu a^\mu} D_L \psi(L^{-1}p)$

irreducible ray representation of the Poincaré group

Hilbert space:

$$\int_{\mathcal{P}_m} \overline{\psi(p)}\psi(p)\,\frac{m\,\mathrm{d}^3p}{p_0} = \int_{\mathcal{P}_m} \psi(p)^{\dagger}\psi(p)\,\frac{m^2\mathrm{d}^3p}{p_0^2}$$

only the solutions of

$$\gamma^{\mu}p_{\mu}\psi(p) = m\psi(p)\ , \qquad (\alpha_j p_j + \beta m)\,\psi(p) = p_0\psi(p)$$

Foldy–Wouthuysen transformation: $\psi^W = \begin{pmatrix} \psi_1^W \\ \psi_2^W \\ 0 \\ 0 \end{pmatrix}$ at any p

$$\psi^W(p) = W(p)\psi(p), \qquad W(p) = \frac{1}{\sqrt{2m(p_0+m)}}\left[(p_0+m)I_4 - \alpha_j p_j\right]$$

Coordinate space:

$$\int \mathrm{e}^{-\mathrm{i}p_\mu x^\mu}\psi(p)\frac{m\mathrm{d}^3p}{p_0} = \int \mathrm{e}^{-\mathrm{i}p_0x_0}\mathrm{e}^{\mathrm{i}p_jx_j}\psi(p)\frac{m\mathrm{d}^3p}{p_0} = (2\pi)^{\frac{3}{2}}\,\psi(t,\mathbf{x})$$

$$\int \left(\psi^\dagger\psi\right)(t,\mathbf{x})\,\mathrm{d}^3x$$

is time independent, equals the momentum space norm

$$\psi(p) \in N_+(p), \qquad \gamma^\mu p_\mu\psi = m\psi, \qquad (\alpha_j p_j + \beta m)\,\psi = p_0\psi$$

$$\Longrightarrow$$

$$\gamma^\mu(\mathrm{i}\partial_\mu)\psi(x) = m\psi(x), \qquad \mathrm{i}\partial_t\psi(t,\mathbf{x}) = \left[\alpha_j\left(-\mathrm{i}\partial_j\right) + \beta m\right]\psi(t,\mathbf{x})$$

Current and Lagrangian:

$\psi^\dagger \psi = \overline{\psi} \beta \psi = \overline{\psi} \gamma^0 \psi$: component of $j^\mu = \overline{\psi} \gamma^\mu \psi$

real, positive definite, conserved, Noether current:

$[\gamma^\mu (\mathrm{i}\partial_\mu) - m]\, \psi = 0$: E–L equation from $\mathcal{L} = \overline{\psi}\, [\gamma^\mu (\mathrm{i}\partial_\mu) - m]\, \psi$,

$\psi \mapsto \mathrm{e}^{\mathrm{i}\chi} \psi$: Noether current

Remark: $\left(\Box + m^2\right) \psi = 0$: Klein–Gordon componentwise

[multiplying by $\gamma^\mu (\mathrm{i}\partial_\mu) + m$]

Spin 0, free quantum mechanics: analogously

$\psi\colon\ \mathbb{C}^4$	$\zeta\colon\ \mathbb{C}^4 \wedge \mathbb{C}^4$
$\psi(p) \in N_+(p)$	$\zeta(p) \in N_+(p) \wedge N_+(p)$
$\gamma^\mu p_\mu \psi = m\psi$	$\gamma^\mu p_\mu \zeta = m\zeta$
$\int \overline{\psi}\psi \frac{m\,\mathrm{d}^3p}{p_0} = \int \psi^\dagger\psi \frac{m^2\mathrm{d}^3p}{p_0^2}$	$\int \mathrm{Tr}\big(\overline{\zeta}\zeta\big) \frac{m\,\mathrm{d}^3p}{p_0} = \int \mathrm{Tr}\big(\zeta^\dagger\zeta\big) \frac{m^2\mathrm{d}^3p}{p_0^2}$
irrep of Poincaré	irrep of Poincaré
essentially 2 DOF	essentially $2 \wedge 2 = 1$ DOF

$$n_1, n_2 \in N_+(p), \quad n_3, n_4 \in N_-(p):$$

$$(\gamma^\mu p_\mu - m)\,(c_{12}\, n_1 \wedge n_2 + c_{13}\, n_1 \wedge n_3 + \cdots + c_{34}\, n_3 \wedge n_4) = 0:$$

only $c_{12} \neq 0$

Foldy–Wouthuysen:

$$W\psi = \begin{pmatrix} \psi_1^W \\ \psi_2^W \\ 0 \\ 0 \end{pmatrix} \qquad W\zeta W^{\mathrm{T}} = \begin{pmatrix} 0 & \zeta_{12}^W & 0 & 0 \\ -\zeta_{12}^W & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$j^\mu = \overline{\psi}\gamma^\mu\psi$$

$$\mathcal{L} = \overline{\psi}\left[\gamma^\mu(\mathrm{i}\partial_\mu) - m\right]\psi$$

KG componentwise

$$j^\mu = \mathrm{Tr}\left(\overline{\zeta}\gamma^\mu\zeta\right)$$

$$\mathcal{L} = \mathrm{Tr}\left\{\overline{\zeta}\left[\gamma^\mu(\mathrm{i}\partial_\mu) - m\right]\zeta\right\}$$

KG componentwise

Nonfree quantum mechanics:

$$\partial_\mu \rightsquigarrow \partial_\mu + \mathrm{i}eA_\mu$$

$$\partial_\mu \operatorname{Tr}\left(\overline{\zeta}_1 \gamma^\mu \zeta_2\right) = 0 \quad \Longrightarrow \quad \partial_t \int \operatorname{Tr}\left(\overline{\zeta}_1 \gamma^0 \zeta_2\right) \equiv \partial_t \int \operatorname{Tr}\left(\zeta_1^\dagger \zeta_2\right) = 0:$$

time independent Hilbert space scalar product

$$\left[\left(\partial^\mu + \mathrm{i}eA^\mu\right)\left(\partial_\mu + \mathrm{i}eA_\mu\right) - m^2\right]\zeta \neq 0 \qquad (!)$$

For the future:

relationship to KG (free, nonfree)

Coulomb problem

second quantization

new types of field theoretical interaction?

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THANK YOU FOR YOUR ATTENTION!