Equations for Heat Transfer

for use in Heat Transfer (BMEGEENAEHK) course in undergraduate mechanical engineering program at BME Faculty of Mechanical Engineering. This document may be used in all quizzes, tests and exams.

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1. Equations you need to know

– Fourier's Law (heat conduction)
– Newton's Law (heat convection)
– Stefan-Boltzmann Law (thermal radiation)
– Equation that defines the thermal resistance
– Equation that defines the overall heat transfer coefficient
– Equation that defines the fin efficiency
– Energy balance of heat exchangers
– Definition of the effectiveness of heat exchangers

2. Steady-state 1D heat conduction

2.1 Thermal resistances

- plane wall $$R_{plane} = \frac{\delta}{A \cdot k}$$
- cylindrical wall $$R_{cylinder} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2 \cdot \pi \cdot L \cdot k}, \quad r_2 > r_1$$
- spherical wall $$R_{sphere} = \frac{\frac{1}{r_1} - \frac{1}{r_2}}{4 \cdot \pi \cdot k}, \quad r_2 > r_1$$
- heat convection from an isothermal surface $$R_{convection} = \frac{1}{h \cdot A_s}$$

2.2 Fins with uniform cross-sectional area

- differential equation

$\frac{d^2\theta(x)}{dx^2} = m^2 \cdot \theta(x)$, where $\theta(x) = T(x) - T_\infty$, $m = \sqrt{\frac{h \cdot P}{k \cdot A}}$, P is the perimeter and A is the cross-sectional area of the fin

- general solution of the differential equation

$$\theta(x) = C_1 \cdot e^{m \cdot x} + C_2 \cdot e^{-m \cdot x}$$

- Solutions with various boundary conditions

The first boundary condition (BC 1) is the same for all cases: $\theta(x=0)=\theta_0$.

Depending on the second boundary condition (BC 2) the solutions and the fin heat transfer rate equations are listed in Table 1 and 2, respectively.

Table 1. Temperature distribution equations for fins with uniform cross-sectional area

CASE	BC 2	$\theta(x)=$
A	$\theta(x=L)=0\,°\mathrm{C}$	$\theta_0 \cdot e^{-mx}$
B	adiabatic tip $\left.\dfrac{d\theta(x)}{dx}\right\|_{x=L}=0$	$\theta_0 \cdot \dfrac{\cosh\left[m(L-x)\right]}{\cosh(mL)}$
C	defined tip temperature $\theta(x=L)=\theta_L$	$\theta_0 \cdot \left[\dfrac{\theta_L/\theta_0 \cdot \sinh(m\cdot x)+\sinh(m\cdot(L-x))}{\sinh(m\cdot L)}\right]$
D	convection from the tip $-k\cdot A\cdot\left.\dfrac{d\theta(x)}{dx}\right\|_{x=L}=h\cdot A\cdot\theta_L$	$\theta_0 \cdot \left[\dfrac{\cosh(m\cdot(L-x))+\left(\dfrac{h}{k\cdot m}\right)\cdot\sinh(m\cdot(L-x))}{\cosh(m\cdot L)+\left(\dfrac{h}{k\cdot m}\right)\cdot\sinh(m\cdot L)}\right]$

Table 2. Heat transfer rate equations for fins with uniform cross-sectional area

CASE	BC 2	$Q_f=$
A	$\theta(x=L)=0\,°\mathrm{C}$	$\sqrt{k\cdot A\cdot h\cdot P}\cdot\theta_0$
B	adiabatic tip $\left.\dfrac{d\theta(x)}{dx}\right\|_{x=L}=0$	$\sqrt{k\cdot A\cdot h\cdot P}\cdot\theta_0\cdot\tanh(m\cdot L)$
C	defined tip temperature $\theta(x=L)=\theta_L$	$\sqrt{k\cdot A\cdot h\cdot P}\cdot\theta_0\cdot\dfrac{\cosh(m\cdot L)-\theta_L/\theta_0}{\sinh(m\cdot L)}$
D	convection from the tip $-k\cdot A\cdot\left.\dfrac{d\theta(x)}{dx}\right\|_{x=L}=h\cdot A\cdot\theta_L$	$\sqrt{k\cdot A\cdot h\cdot P}\cdot\theta_0\cdot\left[\dfrac{\sinh(m\cdot L)+\left(\dfrac{h}{k\cdot m}\right)\cdot\cosh(m\cdot L)}{\cosh(m\cdot L)+\left(\dfrac{h}{k\cdot m}\right)\cdot\sinh(m\cdot L)}\right]$

- thermal resistance of a fin

$$R_{fin}=\frac{1}{\eta_{fin}\cdot h\cdot(P\cdot L+A)}$$

3. The heat diffusion equation for homogeneous solids

- Cartesian coordinate system

$$\frac{\partial}{\partial x}\left(k \cdot \frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k \cdot \frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k \cdot \frac{\partial T}{\partial z}\right)+q_V=\rho \cdot c_p \cdot \frac{\partial T}{\partial t}$$

- Cylindrical coordinate system

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(k \cdot r \cdot \frac{\partial T}{\partial r}\right)+\frac{1}{r^2} \cdot \frac{\partial}{\partial \varphi}\left(k \cdot \frac{\partial T}{\partial \varphi}\right)+\frac{\partial}{\partial z}\left(k \cdot \frac{\partial T}{\partial z}\right)+q_V=\rho \cdot c_p \cdot \frac{\partial T}{\partial t}$$

- Spherical coordinate system

$$\frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(k \cdot r^2 \cdot \frac{\partial T}{\partial r}\right)+\frac{1}{r^2 \cdot \sin^2 \gamma} \cdot \frac{\partial}{\partial \varphi}\left(k \cdot \frac{\partial T}{\partial \varphi}\right)+\frac{1}{r^2 \cdot \sin \gamma} \cdot \frac{\partial}{\partial \gamma}\left(k \cdot \sin \gamma \cdot \frac{\partial T}{\partial \gamma}\right)+q_V=\rho \cdot c_p \cdot \frac{\partial T}{\partial t}$$

4. Lumped capacitance method

The method is applicable if $Bi < 0.1$!

- differential equation

$$-h \cdot A \cdot \theta(t)=c_p \cdot m \cdot \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}$$, where $\theta(t)=T(t)-T_\infty$

- solution

$\theta(t)=\theta_0 \cdot e^{-t/\tau}$, where $\theta_0=\theta(t=0)$ and $\tau=\frac{c_p \cdot m}{h \cdot A_s}$ is the time constant

4. Transient 1D heat conduction

- differential equation

$$\alpha\left(\frac{\partial^2 T}{\partial r^2}+\frac{n}{r} \cdot \frac{\partial T}{\partial r}\right)=\frac{\partial T}{\partial t}$$ or in dimensionless form $$\frac{\partial^2 \vartheta}{\partial \xi^2}+\frac{n}{\xi} \cdot \frac{\partial \vartheta}{\partial \xi}=\frac{\partial \vartheta}{\partial Fo}$$

where for a plane wall n=0, for a cylinder n=1, for a sphere n=2 and $\alpha=\frac{k}{\rho \cdot c_p}$

- one-term analytical solutions

The following one-term analytical solutions are applicable for problems with homogeneous initial conduction and constant and symmetrical boundary conditions of the 1st and 3rd kind. The presented one-term solutions are valid if *Fo*>0.2. The solutions are written in dimensionless form where: $\vartheta = \frac{T - T_\infty}{T_0 - T_\infty}$, $\xi = \frac{r}{L}$, $Fo = \frac{\alpha \cdot t}{L^2}$, $Bi = \frac{h \cdot L}{k}$, $L = \frac{\delta}{2}$ for plane walls, $L = R$ for cylinders and spheres, $E_0 = c_p \cdot m \cdot (T_0 - T_\infty)$ and $\sin(0)/0 = 1$.

	dimensionless temperature	fractional heat loss
plane wall:	$\vartheta(\xi, Fo) = \Psi_1 e^{-\nu_1^2 \cdot Fo} \cos(\nu_1 \cdot \xi)$	$\frac{E}{E_0} = 1 - \vartheta(0, Fo) \frac{\sin \nu_1}{\nu_1}$
cylinder:	$\vartheta(\xi, Fo) = \Psi_1 e^{-\nu_1^2 \cdot Fo} J_0(\nu_1 \cdot \xi)$	$\frac{E}{E_0} = 1 - 2 \cdot \vartheta(0, Fo) \frac{J_1(\nu_1)}{\nu_1}$
sphere:	$\vartheta(\xi, Fo) = \Psi_1 e^{-\nu_1^2 \cdot Fo} \frac{\sin(\nu_1 \cdot \xi)}{\nu_1 \cdot \xi}$	$\frac{E}{E_0} = 1 - 3 \cdot \vartheta(0, Fo) \frac{\sin \nu_1 - \nu_1 \cos \nu_1}{\nu_1^3}$

ψ_1, ν_1 values as a function of the Bi can be found in Table 3, J_0 and J_1 Bessel functions are tabulated in Table 4.

Table 3. ν_1 and Ψ_1 parameters as a function of Bi

	plane wall		cylinder		sphere	
Bi	ν_1	Ψ_1	ν_1	Ψ_1	ν_1	Ψ_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

Table 4. Values of the Bessel function of the first kind (J_0 – zeroth order, J_1 – first order)

z	$J_0(z)$	$J_1(z)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	–0.0968	0.4708
2.8	–0.1850	0.4097
3.0	–0.2601	0.3391
3.2	–0.3202	0.2613

5. Heat exchangers

- notation

  $C = \dot{m} c_p$ – heat capacity rate, W/K

  $C_1 < C_2$

  T' – inlet temperature

  T'' – outlet temperature

  T_1 – temperature of the fluid for which $C_1 < C_2$

  T_2 – temperature of the fluid for which $C_2 > C_1$

  $\theta = \left| T_1 - T_2 \right|$ – temperature difference between the two fluids, °C

  $NTU = \dfrac{U \cdot A_s}{C_1}$ – number of transfer units, –

  $C_R = \dfrac{C_1}{C_2}$ – ratio of the heat capacity rates, –

- logarithmic mean temperature difference (LMTD)

$$\Delta \overline{T_{\ln}} = \frac{\theta_L - \theta_0}{\ln\left(\dfrac{\theta_L}{\theta_0}\right)}$$

- heat transfer rate of parallel and counterflow heat exchangers

$$Q = U \cdot A_s \cdot \Delta \overline{T_{\ln}}$$

  where U is the overall heat transfer coefficient, A_s is the surface area of the heat exchanger

- effectiveness of parallel flow heat exchangers

$$\Phi = \frac{1 - e^{-NTU \cdot (1 + C_R)}}{1 + C_R}$$

- effectiveness of counterflow heat exchangers

$$\Phi = \frac{1 - e^{-NTU \cdot (1 - C_R)}}{1 - C_R \cdot e^{-NTU \cdot (1 - C_R)}} \quad \text{if} \quad C_R \neq 1$$

$$\Phi = \frac{NTU}{1 + NTU} \quad \text{if} \quad C_R = 1$$

6. Heat convection without phase change

6.1 Forced convection

6.1.1 Flat plate in parallel flow

Characteristic length: length of the plate parallel to the flow (L).

Distance from the leading edge: (x).

Characteristic temperature: temperature of the free stream (T_∞).

Other temperature to be known: surface temperature of the plate (T_s).

Characteristic velocity: velocity of the free stream (u_∞).

Dimensionless numbers: $Nu = h \cdot L / k_f$, $Nu_x = h \cdot x / k_f$, $Pr = \nu / \alpha$, $Re = u_\infty \cdot L / \nu$, $Re_x = u_\infty \cdot x / \nu$.

Correction for the temperature dependency of the material properties:

- for liquids $C_T = (Pr / Pr_w)^{0.25}$, where Pr_w is the Prandtl number at T_s,
- for gases $C_T = (T_\infty / T_s)^{0.12}$.

- Isothermal plate in laminar flow:

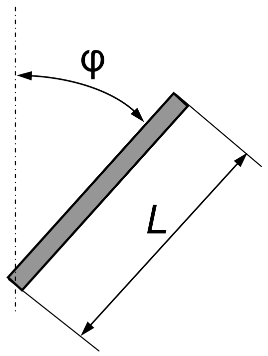
Average Nusselt number:

$$Nu = 0.664 \cdot \sqrt{Re} \cdot Pr^{1/3} \cdot C_T$$

valid if $0.6 \le Pr \le 50$ and $Re < 5 \cdot 10^5$.

6.2 Natural convection

6.2.1 Vertical plate



Characteristic length: length of the plate parallel to the flow (L).

Temperature to be known: surface temperature of the plate (T_s) and the ambient temperature (T_∞).

Characteristic temperature: $T_c = (T_s + T_\infty)/2$

Dimensionless numbers:

$Nu = h \cdot L / k_f$, $Pr = \nu / \alpha$, $Ra = Gr \cdot Pr = \left(g \cdot \cos(\varphi) L^3 \cdot \beta \cdot (|T_s - T_\infty|)\right) / (\nu \cdot \alpha)$.

Accuracy of the calculated heat transfer coefficients: $\pm 20\%$.

laminar flow

Average Nusselt number:

$$Nu = 0.68 + \frac{0{,}670 \cdot Ra^{0{,}25}}{\left[1 + 0{,}671 / Pr^{9/16}\right]^{4/9}} .$$

Valid, if $0 < Ra \le 1 \cdot 10^9$ and $0 \le \varphi \le 60^\circ$.

turbulent flow

Average Nusselt number:

$$Nu = 0.10 \cdot Ra^{1/3} .$$

Valid, if $1 \cdot 10^9 \le Ra \le 1 \cdot 10^{13}$ and $0 \le \varphi \le 60^\circ$.

7. Radiation

7.1 Emitted radiation of the blackbody

- Spectral emissive power (Planck distribution)

$$E_{\lambda,b}(\lambda, T) = \frac{C_1}{\lambda^5 \cdot \left(e^{\frac{C_2}{\lambda \cdot T}} - 1 \right)}$$

where $C_1 = 2 \cdot \pi \cdot h \cdot c_0^2 = 3.742 \cdot 10^8 \ \frac{\text{W} \cdot \mu\text{m}^4}{\text{m}^2}$, $C_2 = \frac{h \cdot c_0}{k} = 1.439 \cdot 10^4 \ \mu\text{m} \cdot \text{K}$, h is the universal Planck constant, k is the universal Boltzmann constant and c_0 is the speed of light in vacuum.

- Wien's Displacement Law

$$\lambda_{max} \cdot T = 2897.8 \ \mu\text{m} \cdot \text{K}$$

- total emissive power

$$E_b(T) = \int_0^\infty E_{\lambda,b}(\lambda, T) \cdot d\lambda = \sigma \cdot T^4$$

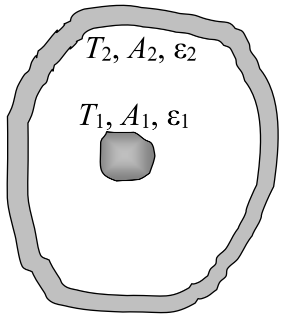
where $\sigma = 5.67 \cdot 10^{-8} \ \frac{\text{W}}{\text{m}^2 \cdot \text{K}^4}$ is the Stefan-Boltzmann constant and T is the absolute temperature.

7.2 Radiation exchange between two diffuse grey objects

- large parallel plates (heat flux)

$$q = \frac{\sigma \cdot \left(T_1^4 - T_2^4\right)}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

- convex object in a large cavity (heat transfer rate)



$$Q = \frac{\sigma \cdot A_1 \cdot \left(T_1^4 - T_2^4\right)}{\frac{1}{\varepsilon_1} + \frac{A_1}{A_2} \cdot \left(\frac{1}{\varepsilon_2} - 1\right)}, \quad A_1 < A_2$$

- general case

$$Q = \sigma \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot F_{1,2} \cdot A_1 \cdot \left(T_1^4 - T_2^4\right)$$

where $F_{1,2}$ is the view factor and $F_{1,2} \cdot A_1 = F_{2,1} \cdot A_2$.

- thermal resistance

$$R_{rad} \cong \frac{1}{\sigma \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot F_{1,2} \cdot A_1 \cdot 4 \cdot T_m^3}$$

where $T_m = \frac{T_1 + T_2}{2}$.